

THE GENERALIZED CHEN'S CONJECTURE ON BIHARMONIC SUBMANIFOLDS IS FALSE

YE-LIN OU* AND LIANG TANG

ABSTRACT

The generalized Chen's conjecture on biharmonic submanifolds asserts that any biharmonic submanifold of a non-positively curved manifold is minimal (see e.g., [CMO1], [MO], [BMO1], [BMO2], [BMO3], [Ba1], [Ba2], [Ou1], [Ou2], [IIU]). In this paper, we prove that this conjecture is false by constructing foliations of proper biharmonic hyperplanes in a 5-dimensional conformally flat space with negative sectional curvature. Many examples of proper biharmonic submanifolds of non-positively curved spaces are also given.

1. BIHARMONIC SUBMANIFOLDS AND THE GENERALIZED CHEN'S CONJECTURE

We work on the category of smooth objects so all manifolds, maps, and tensor fields discussed in this paper are smooth unless there is an otherwise statement.

A **biharmonic map** is a map $\varphi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds which is a local solution of the 4th order PDEs

$$(1) \quad \tau^2(\varphi) := \text{Trace}_g(\nabla^\varphi \nabla^\varphi - \nabla_{\nabla_M}^\varphi) \tau(\varphi) - \text{Trace}_g R^N(d\varphi, \tau(\varphi)) d\varphi = 0,$$

where R^N denotes the curvature operator of (N, h) defined by

$$R^N(X, Y)Z = [\nabla_X^N, \nabla_Y^N]Z - \nabla_{[X, Y]}^N Z,$$

and $\tau(\varphi) = \text{Trace}_g \nabla d\varphi$ is the tension field of φ and $\tau(\varphi) = 0$ means the map φ is harmonic.

Date: 12/31/2010.

1991 Mathematics Subject Classification. 58E20, 53C12, 53C42.

Key words and phrases. Biharmonic maps, biharmonic submanifolds, biharmonic foliations, conformally flat spaces, Chen's conjecture on biharmonic submanifolds.

*Supported by Texas A & M University-Commerce "Faculty Research Enhancement Project" (2009-10) .

Clearly, it follows from (1) that any harmonic map is biharmonic and we call those non-harmonic biharmonic maps **proper biharmonic maps**.

A submanifold M of (N, h) is called a **biharmonic submanifold** if the inclusion map $\mathbf{i} : (M, \mathbf{i}^*h) \rightarrow (N, h)$ is a biharmonic isometric immersion. It is well known that an isometric immersion is minimal if and only if it is harmonic. So a minimal submanifold is trivially biharmonic and we call a non-minimal biharmonic submanifold a **proper biharmonic submanifold**.

Among the fundamental problems in the study of biharmonic maps are the following: **(1) existence problem:** given two model spaces (e.g., some “good” spaces such as spaces of constant sectional curvature or more general symmetric or homogeneous spaces), does there exist a proper biharmonic map from one space into another? **(2) classification problem:** classify all proper biharmonic maps between two model spaces where the existence is known. A typical and challenging classification problem is the following

Chen’s conjecture [CH]: any biharmonic submanifold in a Euclidean space is minimal.

The conjecture was proved to be true for biharmonic surfaces in \mathbb{R}^3 (by Jiang [Ji2] and Chen-Ishikawa [CI] independently) and for biharmonic hypersurfaces in \mathbb{R}^4 ([HV]). Dimitrić [Di] showed that the conjecture is also true for any biharmonic curve, any biharmonic submanifold of finite type, any pseudo-umbilical biharmonic submanifold $M^m \subset \mathbb{R}^n$ with $m \neq 4$, and any biharmonic hypersurface in \mathbb{R}^n with at most two distinct principal curvatures. However, the conjecture is still open in general.

In the same direction of classifying proper biharmonic submanifolds of non-positively curved manifolds, Caddeo, Montaldo and Oniciuc [CMO2] proved that any biharmonic submanifold in hyperbolic 3-space $H^3(-1)$ is minimal, and any pseudo-umbilical biharmonic submanifold $M^m \subset H^n$ with $m \neq 4$ is minimal. It is also shown in [BMO1] that any biharmonic hypersurface of H^n with at most two distinct principal curvatures is minimal. All these results suggest the following generalized Chen’s conjecture on biharmonic submanifolds which was proposed by Caddeo, Montaldo and Oniciuc [CMO1]:

The generalized Chen's conjecture: any biharmonic submanifold of (N, h) with $\text{Riem}^N \leq 0$ is minimal (see e.g., [CMO1], [MO], [BMO1], [BMO2], [BMO3], [Ba1], [Ba2], [Ou1], [Ou2], [IIU]).

The goal of this paper is to prove that the generalized Chen's conjecture for biharmonic submanifolds is false. We accomplished this by using the idea of constructing foliations of proper biharmonic hyperplanes in a conformally flat space given in [Ou1]. The idea is to determine a conformally flat metric on \mathbb{R}^{m+1} so that a foliation by the hyperplanes defined by the graphs of linear functions becomes a proper biharmonic foliation. It turns out that when $m = 4$ the system of biharmonic equations reduces to a single equation which has infinitely many solutions including counter examples to the generalized Chen's conjecture.

2. FOLIATIONS OF CONFORMALLY FLAT SPACES BY BIHARMONIC HYPERPLANES

As conformally flat spaces are going to play a central role in this paper we summarize, in this section, some basic definitions and the relations between various curvatures of two Riemannian manifolds which are conformally related. Two Riemannian metrics g and \bar{g} on M are *conformally equivalent*, if $\bar{g} = e^{2\sigma}g$ for some function σ on M . A map $\varphi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds is *conformal* if $\varphi^*h = e^{2\sigma}g$ for some function σ on M . Say two Riemannian manifolds (M, g) and (N, h) are conformally diffeomorphic, if there exists a conformal diffeomorphism from one space into another. A Riemannian manifold (M^m, g) is a *conformally flat space* if for any point of M there exists a neighborhood which is conformally diffeomorphic to the Euclidean space \mathbb{R}^m . It is well known that any two-dimensional Riemannian manifold is conformally flat due to the existence of isothermal coordinates. For $m = 3$, (M^m, g) is conformally flat if and only if the Schouten tensor H satisfies $(\nabla_X H)(Y, Z) = (\nabla_Y H)(X, Z)$ for any vector fields X, Y and Z on M . For $m \geq 4$, (M^m, g) is conformally flat if and only if the Weyl curvature vanishes identically. It is easy to see that a space of constant sectional curvature is conformally flat but there exist many conformally flat spaces which are not of constant sectional curvature.

Let ∇, R, Ric, K (respectively $\bar{\nabla}, \bar{R}, \bar{\text{Ric}}, \bar{K}$) denote the Levi-Civita connection, Riemannian curvature, Ricci curvature, and sectional curvature of the Riemannian metric g (respectively $\bar{g} = e^{2\sigma}g$). Then, it is not difficult to check (see also [Ha] and [Wa]) the following relations between the connections and the curvatures of the two Riemannian metrics that are conformally equivalent:

$$(2) \quad \bar{\nabla}_X Y = \nabla_X Y + (X\sigma)Y + (Y\sigma)X - g(X, Y) \operatorname{grad}_g \sigma, \quad \forall X, Y \in TM,$$

$$(3) \quad \begin{aligned} \bar{R}(W, Z, X, Y) = & e^{2\sigma} \{ R(W, Z, X, Y) + g(\nabla_X \nabla \sigma, Z)g(Y, W) \\ & - g(\nabla_Y \nabla \sigma, Z)g(X, W) + g(X, Z)g(\nabla_Y \nabla \sigma, W) \\ & - g(Y, Z)g(\nabla_X \nabla \sigma, W) + [(Y\sigma)(Z\sigma) - g(Y, Z)|\nabla \sigma|^2]g(X, W) \\ & - [(X\sigma)(Z\sigma) - g(X, Z)|\nabla \sigma|^2]g(Y, W) \\ & + [(X\sigma)g(Y, Z) - (Y\sigma)g(X, Z)]g(\nabla \sigma, W) \} \end{aligned}$$

for any $W, Z, X, Y \in TM$.

With respect to local coordinates $\{x_i\}$ and the natural frame $\{\frac{\partial}{\partial x_i} = \partial_i\}$, Equation (3) is equivalent to

$$(4) \quad \begin{aligned} e^{-2\sigma} \bar{R}_{ij\,kl} = & R_{ij\,kl} + g_{il}\sigma_{jk} - g_{ik}\sigma_{jl} + g_{jk}\sigma_{il} - g_{jl}\sigma_{ik} \\ & + (g_{il}g_{jk} - g_{ik}g_{jl})|\nabla \sigma|^2, \end{aligned}$$

where we have used the notation $\sigma_{jl} = \nabla_l \sigma_j - \sigma_l \sigma_j = \nabla_l \nabla_j \sigma - \sigma_l \sigma_j$.

By contracting (4) we have

$$(5) \quad \bar{R}_{jk} = R_{jk} - (n-2)\sigma_{jk} - g_{jk}[\Delta \sigma + (n-2)|\nabla \sigma|^2],$$

where Δ and ∇ denote the Laplacian and the gradient operator defined by the metric g .

Let P be a section spanned by an orthonormal basis X, Y with respect to g (hence $\bar{X} = e^{-\sigma}X, \bar{Y} = e^{-\sigma}Y$ form an orthonormal basis with respect to \bar{g}). Then, we have the relationship between the sectional curvatures with respect to metrics \bar{g} and g given by

$$(6) \quad \begin{aligned} e^{2\sigma} \bar{K}(P) = & K(P) - (g(\nabla_X \nabla \sigma, X) + g(\nabla_Y \nabla \sigma, Y)) \\ & - (|\nabla \sigma|^2 - (X\sigma)^2 - (Y\sigma)^2). \end{aligned}$$

We also need the following theorem which will be used to prove our main theorem about biharmonic hypersurfaces in a conformally flat space.

Theorem 2.1. [Ou1] *Let $\varphi : M^m \longrightarrow N^{m+1}$ be an isometric immersion of codimension-one with mean curvature vector $\eta = H\xi$. Then φ is biharmonic if and only if:*

$$(7) \quad \begin{cases} \Delta_g H - H|A|^2 + H \operatorname{Ric}^N(\xi, \xi) = 0, \\ 2A(\operatorname{grad}_g H) + \frac{m}{2} \operatorname{grad}_g H^2 - 2H(\operatorname{Ric}^N(\xi))^\top = 0, \end{cases}$$

where Δ_g and grad_g are the Laplacian and the gradient operators of the hypersurface, and $\text{Ric}^N : T_q N \rightarrow T_q N$ denotes the Ricci operator of the ambient space defined by $\langle \text{Ric}^N(Z), W \rangle = \text{Ric}^N(Z, W)$ and A is the shape operator of the hypersurface with respect to the unit normal vector ξ .

Now we are ready to prove one of the main theorems of this paper.

Theorem 2.2. *For positive integer $m \geq 2$, let $a_i, i = 1, 2, \dots, m$ and c be constants. Then, the isometric immersion $\varphi : \mathbb{R}^m \rightarrow (\mathbb{R}^{m+1}, h = f^{-2}(z)(\sum_{i=1}^m dx_i^2 + dz^2))$, $\varphi(x_1, \dots, x_m) = (x_1, \dots, x_m, \sum_{i=1}^m a_i x_i + c)$ into the conformally flat space is biharmonic if and only if one of the following three cases happens*

- (i) $f' = 0$, in this case φ is minimal (actually, totally geodesic), or
- (ii) $m = 4$ and f is a solution of the equation

$$(8) \quad \sum_{i=1}^4 a_i^2 f^2 f''' + (4 - \sum_{i=1}^4 a_i^2) f f' f'' - 4(2 + \sum_{i=1}^4 a_i^2) (f')^3 = 0,$$

or

- (iii) $a_i = 0$ for $i = 1, \dots, m$ and $f(z) = \frac{1}{Az+B}$, where A and B are constants. In this case each hyperplane is a proper biharmonic hypersurface. This recovers a result (Theorem 3.1) obtained earlier in [Ou1].

Proof. Using the notations $\partial_i = \frac{\partial}{\partial x_i}, i = 1, 2, \dots, m, \partial_{m+1} = \frac{\partial}{\partial z}$, we can easily check that $\{\bar{e}_\alpha = f(z)\partial_\alpha, \alpha = 1, 2, \dots, m+1\}$ constitute an orthonormal frame on the conformally flat space (\mathbb{R}^{m+1}, h) . One can also check that

$$(9) \quad \begin{cases} \varphi(\partial_i) = \frac{1}{f}(\bar{e}_i + a_i \bar{e}_{m+1}), & i = 1, 2, \dots, m; \\ \eta = \frac{1}{f}(\sum_{j=1}^m a_j \bar{e}_j - \bar{e}_{m+1}) \end{cases}$$

constitute a natural frame adapted to the hypersurface with η being a normal vector.

Applying Gram-Schmidt orthonormalization process to the natural frame

$$(10) \quad \begin{cases} \bar{e}_i + a_i \bar{e}_{m+1}, & i = 1, 2, \dots, m; \\ \sum_{j=1}^m a_j \bar{e}_j - \bar{e}_{m+1}, \end{cases}$$

or by a straightforward checking one can verify the following

Claim I. Let $k_i = \frac{1}{\sqrt{1 + \sum_{l=1}^i a_l^2}}, i = 1, \dots, m, k_0 = 1, a_0 = 0$. Then, the vector fields

$$(11) \quad \begin{cases} e_i = -a_i k_i k_{i-1} \sum_{l=1}^{i-1} a_l \bar{e}_l + \frac{k_i}{k_{i-1}} \bar{e}_i + a_i k_i k_{i-1} \bar{e}_{m+1}, & i = 1, 2, \dots, m; \\ e_{m+1} = \sum_{l=1}^m a_l k_m \bar{e}_l - k_m \bar{e}_{m+1}, \end{cases}$$

form an orthonormal frame adapted to the hypersurface $z = \sum_{i=1}^m a_i x_i + c$ with $\xi = e_{m+1}$ being the unit normal vector field.

Let $\bar{h} = \sum_{i=1}^m dx_i^2 + dz^2$ denote the Euclidean metric on \mathbb{R}^{m+1} . Then, $h = e^{-2\sigma} \bar{h}$ with $\sigma = \ln f(z)$. It follows that

$$(12) \quad \text{grad}_h \sigma = \bar{e}_{m+1}(\sigma) \bar{e}_{m+1} = f' \bar{e}_{m+1}.$$

Using the fact that $\bar{\nabla}_{\partial_\alpha} \partial_\beta = 0, \forall \alpha, \beta = 1, 2, \dots, m+1$, and the relation

$$\bar{\nabla}_X Y = \nabla_X Y + (X\sigma)Y + (Y\sigma)X - h(X, Y)\text{grad}_h \sigma,$$

we can compute the Levi-Civita connection ∇ of the conformally flat metric h with respect to the orthonormal frame $\{\bar{e}_i\}$ to get

$$(13) \quad (\nabla_{\bar{e}_\alpha} \bar{e}_\beta) = \begin{pmatrix} f' \bar{e}_{m+1} & 0 & \dots & 0 & -f' \bar{e}_1 \\ 0 & f' \bar{e}_{m+1} & \dots & 0 & -f' \bar{e}_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & f' \bar{e}_{m+1} & -f' \bar{e}_m \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}_{(m+1) \times (m+1)}.$$

A further computation using (13) yields

$$(14) \quad \begin{aligned} \nabla_{e_i} e_i &= k_i^2 k_{i-1}^2 \nabla_{(-a_i \sum_{l=1}^{i-1} a_l \bar{e}_l + \frac{1}{k_{i-1}^2} \bar{e}_i + a_i \bar{e}_{m+1})} (-a_i \sum_{l=1}^{i-1} a_l \bar{e}_l + \frac{1}{k_{i-1}^2} \bar{e}_i + a_i \bar{e}_{m+1}) \\ &= k_i^2 k_{i-1}^2 f' [(a_i^2 \sum_{l=1}^{i-1} a_l^2 + \frac{1}{k_{i-1}^4}) \bar{e}_{m+1} + a_i^2 \sum_{l=1}^{i-1} a_l \bar{e}_l - \frac{a_i}{k_{i-1}^2} \bar{e}_i], \end{aligned}$$

$$(15) \quad \begin{aligned} \nabla_{e_i} e_{m+1} &= \nabla_{(-a_i k_i k_{i-1} \sum_{l=1}^{i-1} a_l \bar{e}_l + \frac{k_i}{k_{i-1}} \bar{e}_i + a_i k_i k_{i-1} \bar{e}_{m+1})} (\sum_{l=1}^m a_l k_m \bar{e}_l - k_m \bar{e}_{m+1}) \\ &= k_m f' [a_i k_i k_{i-1} \bar{e}_{m+1} + \frac{k_i}{k_{i-1}} \bar{e}_i - a_i k_i k_{i-1} \sum_{l=1}^{i-1} a_l \bar{e}_l] \\ &= k_m f' e_i, \end{aligned}$$

and

$$(16) \quad \begin{aligned} \nabla_{e_{m+1}} e_{m+1} &= \nabla_{\sum_{l=1}^m a_l k_m \bar{e}_l - k_m \bar{e}_{m+1}} (\sum_{l=1}^m a_l k_m \bar{e}_l - k_m \bar{e}_{m+1}) \\ &= (1 - k_m^2) f' \bar{e}_{m+1} + \sum_{l=1}^m a_l k_m^2 f' \bar{e}_l. \end{aligned}$$

On the other hand, one can use the relation $e_i = C_i^\alpha \bar{e}_\alpha = f C_i^\alpha \partial_\alpha$, $\xi = C_{m+1}^\alpha \bar{e}_\alpha = f C_{m+1}^\alpha \partial_\alpha$ and the Ricci curvature (5) to verify the following

Claim II. For a hypersurface in the conformally flat space $(\mathbb{R}^{m+1}, h = e^{-2\sigma}(\sum_{i=1}^m dx_i^2 + dz^2))$, we have

$$(17) \quad \text{Ric}(\xi, \xi) = \Delta\sigma + (m-1)[\text{Hess}(\sigma)(\xi, \xi) - (\xi\sigma)^2 + |\text{grad}_h \sigma|^2]$$

$$(18) \quad (\text{Ric}(\xi))^T = (m-1)[\text{grad}_g(\xi\sigma) - \xi(\sigma)\text{grad}_g \sigma + A(\text{grad}_g \sigma)],$$

where grad_g is the gradient defined by the induced metric on the hypersurface.

Substituting

$$\begin{aligned} \Delta\sigma &= \sum_{\alpha=1}^{m+1} [\bar{e}_\alpha \bar{e}_\alpha(\sigma) - (\nabla_{\bar{e}_\alpha} \bar{e}_\alpha)(\sigma)] = f f'' - m(f')^2, \\ \text{Hess}(\sigma)(\xi, \xi) &= e_{m+1} e_{m+1}(\sigma) - (\nabla_{e_{m+1}} e_{m+1})(\sigma) \\ &= k_m^2 f f'' - (1 - k_m^2)(f')^2, \\ \text{grad}_h \sigma &= f' \bar{e}_{m+1}, |\text{grad}_h \sigma|^2 = (f')^2, \quad \xi(\sigma) = -k_m f'. \end{aligned}$$

into (17) we obtain

$$\begin{aligned} (19) \quad \text{Ric}(\xi, \xi) &= \Delta\sigma + (m-1)[\text{Hess}(\sigma)(\xi, \xi) - (\xi\sigma)^2 + |\text{grad}_h \sigma|^2] \\ &= [1 + (m-1)k_m^2] f f'' - m(f')^2. \end{aligned}$$

Noting that $\xi = e_{m+1}$ is the unit normal vector field we can easily compute the components of the second fundamental form to get

$$\begin{aligned} h(e_i, e_i) &= \langle \nabla_{e_i} e_i, e_{m+1} \rangle = -\langle \nabla_{e_i} e_{m+1}, e_i \rangle = -k_m f', \\ h(e_i, e_j) &= \langle \nabla_{e_i} e_j, e_{m+1} \rangle = -\langle \nabla_{e_i} e_{m+1}, e_j \rangle = 0, \quad i \neq j, \end{aligned}$$

from which we conclude that each hyperplane $z = \sum_{i=1}^m a_i x_i + c$ is a totally umbilical hypersurface in the conformally flat space and all principal normal curvature are equal to

$$(20) \quad H = \xi(\sigma) = -k_m f'.$$

It follows that

$$\begin{aligned} (21) \quad [\text{Ric}(\xi)]^T &= (m-1)[\text{grad}_g(\xi\sigma) - \xi(\sigma)\text{grad}_g \sigma + A(\text{grad}_g \sigma)] \\ &= (m-1)\text{grad}_g H, \end{aligned}$$

and the norm of the second fundamental form is given by

$$(22) \quad |A|^2 = \sum_{i=1}^m \langle \nabla_{e_i} \xi, \nabla_{e_i} \xi \rangle^2 = m k_m^2 (f')^2.$$

A further computation gives

$$\begin{aligned}
 e_i(H) &= (-a_i k_i k_{i-1} \sum_{l=1}^{i-1} a_l \bar{e}_l + \frac{k_i}{k_{i-1}} \bar{e}_i + a_i k_i k_{i-1} \bar{e}_{m+1})(-k_m f') \\
 &= -a_i k_i k_{i-1} k_m f f'', \\
 (23) \quad \text{grad}_g H &= \sum_{i=1}^m e_i(H) e_i = - \sum_{i=1}^m a_i k_i k_{i-1} k_m f f'' e_i,
 \end{aligned}$$

$$\begin{aligned}
 (24) \quad e_i e_i(H) &= -e_i(a_i k_i k_{i-1} k_m f f'') \\
 &= -a_i^2 k_i^2 k_{i-1}^2 k_m (f^2 f''' + f f' f''),
 \end{aligned}$$

$$\begin{aligned}
 (25) \quad (\nabla_{e_i} e_i)(H) &= -k_i^2 k_{i-1}^2 k_m f' (a_i^2 \sum_{l=1}^{i-1} a_l \bar{e}_l - \frac{a_i}{k_{i-1}^2} \bar{e}_i)(f') \\
 &= a_i^2 k_i^2 k_{i-1}^2 k_m f f' f''
 \end{aligned}$$

and

$$\begin{aligned}
 (26) \quad \triangle^M H &= \sum_{i=1}^m [e_i e_i(H) - (\nabla_{e_i}^M e_i)(H)] \quad (\text{and by Gauss formula}) \\
 &= \sum_{i=1}^m [e_i e_i(H) - (\nabla_{e_i} e_i)(H) + h(e_i, e_i) \xi(H)] \\
 &= - \sum_{i=1}^m a_i^2 k_i^2 k_{i-1}^2 k_m [(2-m) f f' f'' + f^2 f'''] \\
 &= -(1 - k_m^2) k_m [(2-m) f f' f'' + f^2 f'''],
 \end{aligned}$$

where in obtaining the last equality we have used the identity

$$(27) \quad \sum_{i=1}^m a_i^2 k_i^2 k_{i-1}^2 = (1 - k_m^2)$$

which can be proved by mathematical induction on $m \geq 2$.

Substituting (19), (20), (21), (22), (23), (26), and $A(\text{grad}_g H) = H \text{grad}_g H$ into biharmonic equation (7) we conclude that the isometric immersion φ is biharmonic if and only if

$$(28) \quad \begin{cases} -(1 - k_m^2) f^2 f''' - [3 - m + (2m - 3) k_m^2] f f' f'' + m(1 + k_m^2) (f')^3 = 0, \\ (m - 4) f f' f'' a_i = 0, \quad i = 1, 2, \dots, m. \end{cases}$$

The second equation of (28), and hence the system (28) itself, can be solved by considering the following three cases.

Case 1. $f' = 0$ (which implies $H = -k_m f' = 0$) gives the trivial solution. In this case, φ is actually totally geodesic since its image is a hyperplane in a space that is homothetic to a Euclidean space.

Case 2. $a_i f'' = 0, i = 1, 2, \dots, m$ (which, together with (23), implies $\text{grad}_g H = 0$). In this case the first equation of (28), with the aid of $\Delta^M H = 0$ and (26), can be reduced to

$$(29) \quad [1 + (m-1)k_m^2]ff'' - m(1 + k_m^2)(f')^2 = 0.$$

If $ff'' = 0$, then Equation (29) reduces to $f' = 0$, which gives the trivial solution again, i.e., the hypersurfaces are minimal. If $ff'' \neq 0$, then all $a_i = 0, i = 1, 2, \dots, m$, and hence $k_m = 1$. Thus, Equation (29) reduces to $ff'' - 2(f')^2 = 0$ which has solutions $f(z) = \frac{1}{Az+B}$, where A, B are constants.

Case 3. $m = 4$. In this case, the biharmonic equation (28) reduces to

$$-(1 - k_4^2)f^2f''' - (5k_4^2 - 1)ff'f'' + 4(1 + k_4^2)(f')^3 = 0,$$

from which we obtain Equation (8).

Summarizing the above results we obtain the theorem. □

3. THE GENERALIZED CHEN'S CONJECTURE ON BIHARMONIC SUBMANIFOLDS IS FALSE

In this section, we will show that Equation (8) has many solutions including counter examples to the generalized Chen's conjecture.

Lemma 3.1. *Let $A > 0$, $B > 0$, c be constants, $\mathbb{R}_+^5 = \{(x_1, \dots, x_4, z) \in \mathbb{R}^5 : z > 0\}$ be the upper-half space, and $f : \mathbb{R}_+^5 \rightarrow \mathbb{R}$, $f(x_1, \dots, x_4, z) = f(z) = (Az + B)^t$. Then, for any $t \in (0, 1/2)$ and any $(a_1, a_2, a_3, a_4) \in S^3 \left(\sqrt{\frac{2t}{1-2t}} \right)$, the isometric immersion*

$$(30) \quad \varphi : \mathbb{R}^4 \rightarrow \left(\mathbb{R}_+^5, h = f^{-2}(z) \left[\sum_{i=1}^4 dx_i^2 + dz^2 \right] \right)$$

with $\varphi(x_1, \dots, x_4) = (x_1, \dots, x_4, \sum_{i=1}^4 a_i x_i + c)$ is proper biharmonic into the conformally flat space.

Proof. We try to find special solutions of Equation (8) which have the form $f(z) = (Az + B)^t$. In this case, we have $f' = tA(Az + B)^{t-1}$, $f'' = t(t-1)A^2(Az +$

$B)^{t-2}$, $f''' = t(t-1)(t-2)A^3(Az+B)^{t-3}$. Substituting these into Equation (8) and using the assumption that $A, B > 0$ we have

$$(t-1)(t-2) \sum_{i=1}^4 a_i^2 + (4 - \sum_{i=1}^4 a_i^2)t(t-1) - 4t^2(2 + \sum_{i=1}^4 a_i^2) = 0,$$

which is equivalent to

$$(31) \quad \sum_{i=1}^4 a_i^2 = \frac{2t}{1-2t}.$$

Solving the inequality $\frac{2t}{1-2t} > 0$ we conclude that for any $t \in (0, 1/2)$ and $(a_1, a_2, a_3, a_4) \in S^3\left(\sqrt{\frac{2t}{1-2t}}\right)$ will solve the equation (31) and hence the biharmonic equation (8). From this we obtain the lemma. \square

Example 1. Let $A > 0$, $B > 0$, c be constants and let $t = \frac{1}{6}$ and $(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}) \in S^3\left(\frac{\sqrt{2}}{2}\right)$. Then, by Lemma 3.1, we have a proper biharmonic isometric immersion $\varphi : \mathbb{R}^4 \rightarrow (\mathbb{R}_+^5, h = (Az+B)^{-\frac{1}{3}}(\sum_{i=1}^4 dx_i^2 + dz^2))$ with $\varphi(x_1, \dots, x_4) = (x_1, \dots, x_4, \frac{\sqrt{2}}{4}(x_1 + x_2 + x_3 + x_4) + c)$.

Lemma 3.2. *For constant $A > 0$, $B > 0$ and $t \in (0, 1)$, the conformally flat space $(\mathbb{R}_+^5, h = (Az+B)^{-2t}(\sum_{i=1}^4 dx_i^2 + dz^2))$ has negative sectional curvature.*

Proof. Let $f(z) = (Az+B)^t$. Then, as in the proof of Theorem 2.2, we use $\bar{e}_i = f(z)\partial_i, i = 1, \dots, 5$ to denote the orthonormal frame on $(\mathbb{R}_+^5, h = (Az+B)^{-2t}(\sum_{i=1}^4 dx_i^2 + dz^2))$. Let P be a plane section at any point and suppose that P is spanned by an orthonormal basis X, Y . Then, we have $X = \sum_{i=1}^5 a_i \bar{e}_i, Y = \sum_{i=1}^5 b_i \bar{e}_i$. Using sectional curvature relation (6) and the fact that the sectional curvature $\bar{K}(p)$ of $(\mathbb{R}_+^5, \bar{h} = \sum_{i=1}^4 dx_i^2 + dz^2)$ vanishes identically we find the sectional curvature of the conformally flat space to be

$$\begin{aligned} K(P) &= (h(\nabla_X \nabla \sigma, X) + h(\nabla_Y \nabla \sigma, Y)) + (|\nabla \sigma|^2 - (X\sigma)^2 - (Y\sigma)^2) \\ &= X(X\sigma) + Y(Y\sigma) - (\nabla_X X)(\sigma) - (\nabla_Y Y)(\sigma) + (|\nabla \sigma|^2 - (X\sigma)^2 - (Y\sigma)^2), \end{aligned}$$

where $\sigma = \ln f(z)$. A straightforward computation gives

$$\begin{aligned}
X\sigma &= \sum_{i=1}^5 a_i \bar{e}_i(\sigma) = a_5 f', \\
X(X\sigma) &= \sum_{i=1}^5 a_i \bar{e}_i(a_5 f') = \sum_{i=1}^5 a_i \bar{e}_i(a_5) f' + a_5^2 f f'', \\
\nabla_X X &= \nabla_{\sum_{i=1}^5 a_i \bar{e}_i} \left(\sum_{j=1}^5 a_j \bar{e}_j \right) \\
&= \sum_{i=1}^5 a_i \bar{e}_i \left(\sum_{j=1}^5 a_j \right) \bar{e}_j + \sum_{i=1}^4 a_i^2 f' \bar{e}_5 - \sum_{i=1}^4 a_i a_5 f' \bar{e}_i \\
(\nabla_X X)(\sigma) &= \sum_{i=1}^5 a_i \bar{e}_i(a_5) f' + \sum_{i=1}^4 a_i^2 (f')^2. \\
X(X\sigma) - (\nabla_X X)(\sigma) &= a_5^2 f f'' - \sum_{i=1}^4 a_i^2 (f')^2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
Y(Y\sigma) &= b_5 f', (\nabla_Y Y)(\sigma) = \sum_{i=1}^5 b_i \bar{e}_i(b_5) f' + \sum_{i=1}^4 b_i^2 (f')^2 \\
Y(Y\sigma) - (\nabla_Y Y)(\sigma) &= b_5^2 f f'' - \sum_{i=1}^4 b_i^2 (f')^2,
\end{aligned}$$

from which we have

$$\begin{aligned}
K(P) &= X(X\sigma) + Y(Y\sigma) - (\nabla_X X)(\sigma) - (\nabla_Y Y)(\sigma) + (|\nabla\sigma|^2 - (X\sigma)^2 - (Y\sigma)^2) \\
&= (a_5^2 + b_5^2) f f'' - (f')^2.
\end{aligned}$$

For $f = (Az + B)^t$, we have $f' = tA(Az + B)^{t-1}$, $f'' = t(t-1)A^2(Az + B)^{t-2}$, so

$$\begin{aligned}
K(P) &= (a_5^2 + b_5^2) f f'' - (f')^2 \\
&= A^2(Az + B)^{t-2} [(a_5^2 + b_5^2) t(t-1) - t^2],
\end{aligned}$$

which is strictly negative since $[(a_5^2 + b_5^2) t(t-1) - t^2] < 0$ for $0 < t < 1$, and $A^2(Az + B)^{t-2} > 0$ for $z > 0$. From this we obtain the Lemma. \square

Combining Lemma 3.1 and Lemma 3.2 we have

Theorem 3.3. *Let $A > 0$, $B > 0$, c be constants, $\mathbb{R}_+^5 = \{(x_1, \dots, x_4, z) \in \mathbb{R}^5 : z > 0\}$ be the upper-half space, and $f : \mathbb{R}_+^5 \rightarrow \mathbb{R}$, $f(z) = (Az + B)^t$. Then, for any $t \in (0, 1/2)$ and any $(a_1, a_2, a_3, a_4) \in S^3 \left(\sqrt{\frac{2t}{1-2t}} \right)$, the isometric immersion*

$$(32) \quad \varphi : \mathbb{R}^4 \longrightarrow \left(\mathbb{R}_+^5, h = f^{-2}(z) \left[\sum_{i=1}^4 dx_i^2 + dz^2 \right] \right)$$

with $\varphi(x_1, \dots, x_4) = (x_1, \dots, x_4, \sum_{i=1}^4 a_i x_i + c)$ gives a proper biharmonic hypersurface into the conformally flat space with strictly negative sectional curvature. These provide infinitely many counter examples to the generalized Chen's conjecture on biharmonic submanifolds.

The following corollary can be used to construct proper biharmonic submanifolds of any codimension in a nonpositively curved manifold.

Corollary 3.4. *For any positive integer k , there exists a proper biharmonic submanifold of codimension k in a nonpositively curved space. Thus, the generalized Chen's conjecture is false.*

Proof. Let

$$(33) \quad \varphi : \mathbb{R}^4 \longrightarrow \left(\mathbb{R}_+^5, h = f^{-2}(z) \left[\sum_{i=1}^4 dx_i^2 + dz^2 \right] \right)$$

with $\varphi(x_1, \dots, x_4) = (x_1, \dots, x_4, \sum_{i=1}^4 a_i x_i + c)$ be one of the proper biharmonic hypersurface given in Theorem 3.3 and $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^{k-1} \equiv (\mathbb{R}^{n+k-1}, h_0)$ with $\psi(y) = (y, 0)$ be the totally geodesic embedding of a subspace into a Euclidean space. Then, the isometric embedding $\phi : \mathbb{R}^4 \times \mathbb{R}^n \rightarrow (\mathbb{R}_+^5 \times \mathbb{R}^{n+k-1}, h + h_0)$ with $\phi(x, y) = (\varphi(x), \psi(y))$ gives a submanifold of codimension k . Since ϕ is biharmonic with respect to each variable separately and it is proper biharmonic with respect to x -variable by Theorem 3.3, we can use Proposition 2.1 in [Ou2] to conclude that ϕ is a proper biharmonic embedding. Thus, the image of ϕ provides a proper biharmonic submanifold of codimension k . Since, by Lemma 3.2, the conformally flat space $(\mathbb{R}_+^5, h = (Az + B)^{-2t} (\sum_{i=1}^4 dx_i^2 + dz^2))$ has negative sectional curvature and the Euclidean space $(\mathbb{R}^{n+k-1}, h_0)$ has zero curvature, their product $(\mathbb{R}_+^5 \times \mathbb{R}^{n+k-1}, h + h_0)$ gives a space of nonpositive curvature. Thus, we obtain the corollary. \square

Acknowledgments. Both authors are grateful to Cezar Oniciuc for a careful reading and checking the computations in the paper. We especially want to thank him for pointing out a mistake in the computation of the Laplacian of the mean curvature of the hypersurface in the previous version.

REFERENCES

- [Ba1] A. Balmus, *Perspectives on biharmonic maps and submanifolds*, Differential geometry, Proceedings of the VIII International Colloquium, edited by Jesús A. Alvarez López, Eduardo Garcia-Rio, 257–265, World Sci. Publ., Hackensack, NJ, 2009.
- [Ba2] A. Balmus, *Biharmonic maps and submanifolds*, Balkan Society of Geometers, Differential Geometry - Dynamical Systems * Monographs, 2009.
- [BMO1] A. Balmus, S. Montaldo and C. Oniciuc, *Classification results for biharmonic submanifolds in spheres*, Israel J. Math. 168 (2008), 201–220.
- [BMO2] A. Balmus, S. Montaldo, C. Oniciuc, *Biharmonic submanifolds in sphere forms*, Symposium Valenceiennes (2008), 25–32.
- [BMO3] A. Balmus, S. Montaldo, C. Oniciuc, *Classification results and new examples of proper biharmonic submanifolds in spheres*, Note di Matematica, Note Mat. 1(2008), suppl. n. 1, 49–61.
- [CMO1] R. Caddeo, S. Montaldo, and C. Oniciuc, *Biharmonic submanifolds of S^3* . Internat. J. Math. 12 (2001), no. 8, 867–876.
- [CMO2] R. Caddeo, S. Montaldo and C. Oniciuc, *Biharmonic submanifolds in spheres*, Israel J. Math. 130 (2002), 109–123.
- [CH] B. Y. Chen, *Some open problems and conjectures on submanifolds of finite type*, Soochow J. Math. 17 (1991), no. 2, 169–188.
- [CI] B. Y. Chen and S. Ishikawa, *Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces*, Kyushu J. Math. 52 (1998), no. 1, 167–185.
- [Di] I. Dimitrić, *Submanifolds of E^m with harmonic mean curvature vector*, Bull. Inst. Math. Acad. Sinica 20 (1992), no. 1, 53–65.
- [Ha] L. Habermann, *Riemannian metrics of constant mass and moduli spaces of conformal structures*, Lecture Notes in Mathematics, 1743. Springer-Verlag, Berlin, 2000.
- [HV] T. Hasanis and T. Vlachos, *Hypersurfaces in E^4 with harmonic mean curvature vector field*, Math. Nachr. 172 (1995), 145–169.
- [IIU] T. Ichiyama, J. Inoguchi and H. Urakawa, *Classifications and Isolation Phenomena of Bi-Harmonic Maps and Bi-Yang-Mills Fields*, arXiv:0912.4806, Preprint, 2009.
- [In] J. Inoguchi, *Submanifolds with harmonic mean curvature vector field in contact 3-manifolds*. Colloq. Math. 100(2004), no. 2, 163–179.
- [Ji1] G. Y. Jiang, *2-Harmonic maps and their first and second variational formulas*, Chin. Ann. Math. Ser. A 7(1986) 389–402.
- [Ji2] G. Y. Jiang, *Some non-existence theorems of 2-harmonic isometric immersions into Euclidean spaces*, Chin. Ann. Math. Ser. 8A (1987) 376–383.
- [MO] S. Montaldo and C. Oniciuc, *A short survey on biharmonic maps between Riemannian manifolds*, Rev. Un. Mat. Argentina 47 (2006), no. 2, 1–22 (2007).
- [Ou1] Y. -L. Ou, *Biharmonic hypersurfaces in Riemannian manifolds*, Pacific J. of Math. 248 (1), (2010), 217–232.
- [Ou2] Y. -L. Ou, *Some constructions of biharmonic maps and Chen's conjecture on biharmonic hypersurfaces*, arXiv:0912.1141, preprint 2009.
- [Wa] G. Walschap, *Metric structures in differential geometry*, Graduate Texts in Mathematics, 224. Springer-Verlag, New York, 2004.

DEPARTMENT OF MATHEMATICS,
TEXAS A & M UNIVERSITY-COMMERCE,
COMMERCE TX 75429,
USA.
E-MAIL: YELIN_OU@TAMU-COMMERCE.EDU (Ou),

COLLEGE OF MATHEMATICS & COMPUTER SCIENCES
GUANGXI UNIVERSITY FOR NATIONALITIES
NANNING 530006,
P. R. CHINA